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Dimensional analysis in physics and the Buckingham theorem

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Abstract

Dimensional analysis is a simple, clear and intuitive method for determining the functional dependence of physical quantities that are of importance to a certain process. However, in physics textbooks, very little space is usually given to this approach and it is often presented only as a diagnostic tool used to determine the validity of dependences otherwise obtained. This paper presents the basics of dimensional analysis in two cases: the resistance force of the fluid that occurs when a body moves through it and the speed of propagation of waves on water. After that, a general approach to dimensional analysis based on the Buckingham theorem is shown. The material presented in the paper could be useful to both students of physics and physics graduates.

1. Introduction

As is known, physical quantities may have dimensions or are dimensionless. If the quantity has a dimension, then its numeric value will depend on the choice of system of measuring units⁴, and if it is dimensionless, it will not⁵. The dimension of the physical quantity indicates its physical nature, because regardless of whether we express the measured distance in feet or metres, it is a measurement of length. The symbols that are commonly used to designate the dimensions of basic physical quantities such as length, mass and time are L , M and T .⁶ Base physical quantities by mutual multiplication and division give new, derived physical quantities

⁴ For example, the time interval between two successive sunrises can be expressed as 1 day, as 24 h, as 1440 min or 86 400 s. We see that the numeric value changes depending on the unit of time, although it is always one and the same interval of time.

⁵ Height of Mount Everest ($h = 8.848$ km) and the radius of the Earth ($R = 6370$ km) are obviously dimensional quantities, but their ratio, $h/R = 0.0014$, is dimensionless, and therefore, independent of the unit system.

⁶ The dimension of volume of electric current is denoted by I , and the dimension of absolute temperature by Θ .

with corresponding dimensions. Thus the ratio of the travelled distance and the time interval gives a new physical quantity (speed), whose dimension is L/T .

Physical law and formula, by which it is expressed, must not depend on the unit system. It is quite natural because the laws of nature establish a link between the quantities, while the unit system in which they are expressed is a matter of people's agreement. Like any agreement, this one is also subject to changes, so before today's approved, and more or less generally accepted International System of Units (SI), we had different measuring unit systems throughout history. A very important, though seemingly trivial, conclusion follows that both sides of any equation between physical quantities must have the same dimensions. The procedure of analysing dimensions of physical quantities, starting from the equality of left- and right-side dimensions of the equation which connects them, is called dimensional analysis [1, 2].

After some time, it turned out that dimensional analysis is also a method for reducing the complex dependence of the physical quantity to its simplest (most economical) form which is subsequently theoretically or experimentally analysed [3]. Areas in which it is successfully applied are difficult to enumerate. Some of them are astrophysics, electrodynamics, aerodynamics, design and construction of ships, mass and heat transfer, mechanics of elastic and plastic structures, simulation of nuclear reactors, biology and economy [4–6]. Dimensional analysis is particularly useful in studying the new phenomena for which the appropriate equations and boundary conditions are not yet fully known.

The contents of the paper may be used at different levels of complexity. The two explicit examples presented in section 2 can be useful to students, in the analysis of movement of the body through a fluid and the propagation of waves on water, for better understanding of the essence of the terms which describe these processes. The Buckingham theorem, presented in section 3, indicates the tremendous importance of dimensional analysis for understanding the nature of physical quantities and their mutual relations shown by physical laws. This and the next two sections are a bit more mathematically demanding and thus accessible to senior students and graduate students of physics. Sections 4 and 5 describe the application of the Buckingham theorem to examples discussed in section 2.

2. Intuitive approach to dimensional analysis

When we want to show the dimension of a physical quantity we use square brackets $[]$. If we are interested in the dimensions of speed u , we will write $[u] = L/T$. Dimension of area S is $[S] = L^2$, of volume V is $[V] = L^3$, and acceleration a is $[a] = L/T^2$. As we will see later, when applying dimensional analysis, in principle, two problems may occur. The first concerns the choice of the physical quantities on which the observed quantity depends. Another problem is the existence of a group of quantities whose algebraic combinations form dimensionless factors when expressing the required relations. Let us look at how it works in the cases of two important examples: the resistance of the medium to the movement of the object through it and propagation of waves on water.

2.1. An object moving through the fluid

Let us try, using dimensional analysis, to determine the resistive force which occurs when an object moves through a medium. As already mentioned, it is necessary to determine the quantities on which this force may depend. Our experience says that the resistive force of the medium increases with increasing velocity u . In addition, objects with larger cross-section endure greater resistance than those with smaller cross-section. For this reason, the equation

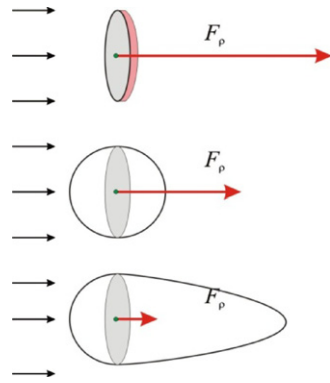


Figure 1. The resistance force of a medium that flows around different shaped objects of equal characteristic dimensions.

(This figure is in colour only in the electronic version)

of force certainly must include velocity u and cross-sectional area of the object S . Finally, the size of the resistance depends on the characteristics of the medium itself. Here is the first problem: which characteristic of the medium do we choose? It seems natural that we should choose the density of fluid ρ , because the denser it is, the more it influences the movement of the object. According to all aforesaid, we will assume the resistance force of the medium in the form of

$$F_\rho = \frac{C}{2} u^\alpha S^\beta \rho^\gamma \quad (1)$$

(multiplier $1/2$ may be included in C , but was separated for historical reasons). As force has dimensions of mass multiplied by acceleration, i.e. $[F] = LT^{-2}M$, from the condition of equality of dimensions on the left- and right-hand side of equation (1), we obtain

$$LT^{-2}M = (LT^{-1})^\alpha (L^2)^\beta (ML^{-3})^\gamma = L^{\alpha+2\beta-3\gamma} T^{-\alpha} M^\gamma.$$

Then follows

$$\begin{aligned} L: \quad & 1 = \alpha + 2\beta - 3\gamma, \\ T: \quad & -2 = -\alpha, \\ M: \quad & 1 = \gamma. \end{aligned}$$

Their solution is $\alpha = 2$, $\beta = 1$ and $\gamma = 1$, so the formula we looked for is

$$F_\rho = CS \frac{\rho u^2}{2}. \quad (2)$$

The value of the coefficient C is determined experimentally. It is shown that it depends on the shape of the object (figure 1), i.e. on the way the fluid will flow around it. So for an object in the shape of a disc, the coefficient C is between 1.1 and 1.2, while for the shape of a ball it is between 0.4 and 2, and for an object in the shape of a drop it is $C \approx 0.04$, or about ten times less than for the ball and about 30 times less than for the disc of the same radius [7, 8].

The formula of the resistive force (2) was obtained under the assumption that the dominant feature of the medium, providing resistance to the object movement through it, is density. But what if, instead of density, the viscosity is taken as a characteristic of the medium, in other

words, the coefficient of viscosity η as the characteristic quantity, the dimension of which is $[\eta] = ML^{-1}T^{-1}$? In this case, we will assume that the formula of force is as follows:

$$F_{\eta} = Bu^{\alpha}S^{\beta}\eta^{\delta}, \quad (3)$$

where B is a constant that, as well as constant C , depends on the shape of the object. Dimensions of equation (3) are

$$LT^{-2}M = (LT^{-1})^{\alpha}(L^2)^{\beta}(ML^{-1}T^{-1})^{\delta} = L^{\alpha+2\beta-\delta}T^{-\alpha-\delta}M^{\delta}.$$

From the corresponding system of equations $\alpha = 1$, $\beta = 1/2$ and $\delta = 1$ are obtained, so the requested formula is

$$F_{\eta} = B\eta u\sqrt{S}. \quad (4)$$

Size \sqrt{S} is proportional to the characteristic object size L so the above formula takes the form of Stokes' law⁷:

$$F_{\eta} = B\eta uL. \quad (5)$$

As we can see, formulae (2) and (5) are quite different: in one of them the dependence of force on the speed is square and in the second it is linear. Which one is correct? To answer this question we would have to, in some way, decide which of the medium characteristics (density or viscosity) dominates in the problem we are solving. It can be concluded that if the dominant feature of the medium is density, formula (2), which represents the resistance force arising from differences in pressure on the front and backside of the object, is true. When the resistance force is the result of friction, i.e. viscosity, formula (5) is valid. The ratio of these two forces, F_{ρ} and F_{η} ,

$$\frac{F_{\rho}}{F_{\eta}} = \frac{C}{2B} \frac{\rho u^2 S}{\eta u L}, \quad (6)$$

taking $S \propto L^2$, can be written as

$$\frac{F_{\rho}}{F_{\eta}} \propto \frac{\rho u L}{\eta}. \quad (7)$$

The dimensionless ratio we obtained is called the Reynolds number

$$Re = \frac{\rho u L}{\eta}, \quad (8)$$

and plays a great role in hydro- and aero-dynamics. The books in the field of fluid dynamics usually state that for the case of $Re < 1$, the resistance of the medium caused by the difference in pressures on the front and backside of the object can be ignored, and the resistive force is equal to the force of viscous friction. Conversely, when the values of the Reynolds number are high, we should consider the force caused by pressure differences and ignore the force of friction. The question is whether it is possible to unify the dependence of resistance force on density and viscosity. This would mean that we should seek the force in the form of

$$F_{\rho,\eta} = \frac{A}{2} u^{\alpha} S^{\beta} \rho^{\gamma} \eta^{\delta} \quad (9)$$

(A is a constant). According to dimensional analysis of the left- and right-hand side, we get $LT^{-2}M = (LT^{-1})^{\alpha}(L^2)^{\beta}(L^{-3}M)^{\gamma}(ML^{-1}T^{-1})^{\delta} = L^{\alpha+2\beta-3\gamma-\delta}T^{-\alpha-\delta}M^{\gamma+\delta}$, and therefore formulae

$$L: \quad 1 = \alpha + 2\beta - 3\gamma - \delta,$$

$$T: \quad -2 = -\alpha - \delta,$$

$$M: \quad 1 = \gamma + \delta.$$

⁷ As is known, if the object is in the shape of a ball with the radius r , then $\sqrt{S} \propto r$ and $B\sqrt{S} = 6\pi r$.

First, it is observed that the number of equations is now smaller than the number of unknowns (three equations and four parameters), which means that one of the unknowns must remain undetermined. If the parameters α , γ and β are expressed through δ ($\gamma = 1 - \delta$, $\alpha = 2 - \delta$, $\beta = 1 - \delta/2$), the formula for the resistive force can be written as

$$F_{\rho,\eta} = \frac{A}{2} u^{2-\delta} S^{1-\frac{\delta}{2}} \rho^{1-\delta} \eta^\delta = A \frac{\rho S u^2}{2} \left(\frac{\sqrt{S} u \rho}{\eta} \right)^{-\delta}. \quad (10)$$

The fact that δ is arbitrary power means that the factor in parentheses from the previous formula has no dimensions and can be included in the dimensionless factor which, in that case, is not constant but becomes a function of the above-mentioned dimensionless parameter. Note that this dimensionless ratio is exactly the Reynolds number, so the resistive force formula can be written as

$$F_{\rho,\eta} = A(Re) S \frac{\rho u^2}{2}, \quad (11)$$

where the function $A(Re)$ is called the coefficient of resistance. The Reynolds number, therefore, plays an important role in determining the character of the resistive force. The exact dependence of the coefficient of resistance on the Reynolds number cannot be determined in this way, but by using different theoretical approaches or by experiment.

2.2. Wave propagation on water

Another interesting example of dimensional analysis application is determining the formula for the speed of surface waves on water. Our first obvious experiences and notions concerning the waves come exactly from this type of wave. They occur if, under the influence of an external action, the free surface of liquid is disturbed from its equilibrium position. The restoring forces that appear during this are gravitational force and force of surface tension. Waves caused in such a way represent a complex movement that is influenced not only by gravity and surface tension, but also by viscosity, depth of liquid, etc. Rough division of waves on water according to the dominant effect of the restoring force, if we neglect waves whose wavelength is significantly greater than the depth of water, could be reduced to

- *gravitational waves* in which the influence of surface tension can be neglected in comparison to the influence of the gravitational field strength and whose wavelength is relatively large, but small compared to the depth of water⁸;
- *capillary waves* where the gravitational influence is negligible compared to the influence of surface tension and whose wavelength is relatively small, and small compared to the depth of water⁹.

Consider, for example, propagation of gravity waves in the ocean. It is reasonable to assume that the speed of propagation of such a wave u_g depends on the wavelength λ , the gravitational field strength g and density of water ρ , so the form is

$$u_g = D \rho^\alpha \lambda^\beta g^\delta \quad (12)$$

(D is a constant). From the condition of equality of dimensions on the left- and right-hand side of the equation, we get

$$LT^{-1} = (ML^{-3})^\alpha (L)^\beta (LT^{-2})^\delta = L^{-3\alpha+\beta+\delta} T^{-2\delta} M^\alpha.$$

⁸ In these waves λ is larger than 20 cm. The tsunami is an example of a wave whose wavelength often exceeds 100 km so they can reach speeds of 600–800 km h⁻¹. They can carry great energy and can cross large distances.

⁹ In these waves λ is smaller than 2 mm. They do not cross great distances and have relatively little energy.

Here it follows that $\alpha = 0$, $\beta = \delta = 1/2$, so the velocity of gravitational waves is

$$u_g = D\sqrt{\lambda g}. \quad (13)$$

Note that in relation (12) there is a possibility that the speed of waves also depends on the density of liquid, but it does not occur in the final formula¹⁰ ($\alpha = 0$), but there is only dependence on the strength of the gravitational field and the wavelength. The correct formula for the speed of gravitational waves [9] is

$$u_g = \frac{1}{\sqrt{2\pi}}\sqrt{\lambda g}. \quad (14)$$

If surface tension is taken as a dominant restoring force, the waves are capillary and their speed u_c will depend on the density ρ , wavelength λ and the coefficient of surface tension σ ¹¹, and will form

$$u_c = E\rho^\alpha\lambda^\beta\sigma^\delta \quad (15)$$

(E is a constant). From the condition of dimension equality of the left- and right-hand side of the equation, we get

$$LT^{-1} = (ML^{-3})^\alpha(L)^\beta(MT^{-2})^\delta = L^{-3\alpha+\beta}T^{-2\delta}M^{\alpha+\delta}.$$

Simultaneous solution of the corresponding system of equations gives $\alpha = \beta = -1/2$, $\delta = 1/2$, and the speed of capillary waves is presented by the relation

$$u_c = E\sqrt{\frac{\sigma}{\lambda\rho}}. \quad (16)$$

Therefore, the speed of capillary waves is inversely proportional to their wavelength, and with increasing wavelength, the speed of these waves, unlike gravitational, decreases. The exact formula for the speed of capillary waves is [9]

$$u_c = \sqrt{2\pi}\sqrt{\frac{\sigma}{\lambda\rho}}. \quad (17)$$

Formulae (13) and (16) for the speed of waves on water are quite different. One can question again the choice of criteria for the application of one or other equation. To answer this question we would have to decide which restoring force (gravitational or surface tension force) dominates in the current problem we are solving. The problem comes down to the introduction of specific criteria for assessing the size of the wavelength. To this end, we will find the ratio between speed u_c and u_g :

$$\frac{u_c}{u_g} = \frac{E\sqrt{\frac{\sigma}{\lambda\rho}}}{D\sqrt{\lambda g}} \propto \sqrt{\frac{\sigma}{\lambda^2 g\rho}}. \quad (18)$$

The dimensionless ratio obtained is associated with the so-called Bond number,

$$Bo = \frac{\lambda^2 g\rho}{\sigma}, \quad (19)$$

which in fluid mechanics gives the relationship of gravity and surface tension forces. At large values of the Bond number, we can neglect the effect of surface tension, while the small value of this number signals its dominant influence. On the other hand, mid-values of the Bond number signal the non-trivial relation between these two influences. Let us try now, as in the

¹⁰ Since density depends on mass and volume, and that the gravitational field of the same force acts on all objects, regardless of their masses, this result is logical.

¹¹ Since the coefficient of surface tension is equal to the work per area unit, its dimensions are $[\sigma] = MT^{-2}$.

previous example, to merge the dependence of the wave propagation speed on surface tension and strength of the gravitational field. This would mean that the speed should be sought in the form

$$u_{g,c} = F\lambda^\alpha \rho^\beta g^\gamma \sigma^\delta \quad (20)$$

(F is a constant). Analysis of the left- and right-hand side dimensions gives

$$LT^{-1} = L^\alpha (ML^{-3})^\beta (LT^{-2})^\gamma (MT^{-2})^\delta = L^{\alpha-3\beta+\gamma} T^{-2\gamma-2\delta} M^{\beta+\delta}$$

where the following equations can be derived:

$$L: \quad 1 = \alpha + 3\beta + \gamma,$$

$$T: \quad -1 = -2\gamma - 2\delta,$$

$$M: \quad 0 = \beta + \delta.$$

The number of equations is again smaller than the number of unknowns, which means that one of the unknowns must remain undetermined. If the parameters α , β and γ are expressed through δ we get $\alpha = 1/2 - 2\delta$, $\beta = -\delta$ and $\gamma = 1/2 - \delta$. The formula for the speed of wave propagation can be written as

$$u_{g,c} = F\sqrt{\lambda g} \left(\frac{\sigma}{\lambda^2 g \rho} \right)^\delta. \quad (21)$$

As δ is an arbitrary power, the factor in parentheses from the previous formula is dimensionless, and it can be included in the dimensionless constant, and all that, in fact, into an arbitrary function of this parameter. As this dimensionless ratio is associated with the Bond number, the formula for the speed of waves can be written as

$$u_{g,c} = F(Bo)\sqrt{\lambda g}. \quad (22)$$

Complex theoretical analysis shows that the speed of waves on the water is represented by the general formula [9]

$$u = \sqrt{\left(\frac{g\lambda}{2\pi} + \frac{2\pi\sigma}{\rho\lambda} \right) \tanh \frac{2\pi h}{\lambda}},$$

where h marks the depth of water. This formula, starting with the fulfilled requirement that $\lambda \approx h(\tan(2\pi h/\lambda) \approx 1)$, includes the two cases discussed. Based on the observed examples, we can note that dimensional analysis of formulae that define dependences of physical quantities relevant for the given process leads to interesting conclusions, but also opens up some new problems. Therefore, we will deal with dimensional analysis in general and apply the results obtained to the two observed examples.

3. Dimensional analysis and Buckingham theorem

The physical law that describes a phenomenon in general can be written in the form of

$$q_0 = f(q_1, q_2, \dots, q_n), \quad (23)$$

where q_0 is a physical quantity whose value is determined by mutually independent¹² physical quantities q_1, q_2, \dots, q_n .¹³ The analytical function that describes the dependence of these

¹² As we will see, it is convenient to distinguish *physically* and *dimensionally* independent physical quantities. In this case, it is independent physical quantities that can be dimensionally dependent (meaning that the dimensions of some of them can be expressed through dimensions of others in this group).

¹³ These physical quantities are actually the data we get in the corresponding experiment.

quantities is marked with f . Set q_1, q_2, \dots, q_n is *complete* if it contains all quantities which affect q_0 and *independent* if any member of the set does not affect the value of other members¹⁴. When it comes to the form of the functional dependence (23) that connects the dependent and independent physical quantities, one should note that it must be *dimensionally homogeneous* [10, 11]. This means that when the basic unit system changes, q_0 and f change by the same factor. This natural requirement is reduced to the following series of restrictions that must apply to the physical law (23):

- both sides of the equation must have the same dimensions;
- if a sum of quantities appears in f , all the terms in the sum must have the same dimensions;
- all arguments of any exponential, logarithmic, trigonometric, or other special functions that appear in f must be dimensionless¹⁵.

An important consequence of dimensional homogeneity is the already highlighted fact that the form of physical laws is independent of the choice of the system of base units (and its size). When the system of base units is selected, it is necessary to express the dimensions of variables q_0 and q_1, q_2, \dots, q_n within it. If we only deal with mechanical processes, as we have seen, the dimensions of each variable can be written in the form

$$[q_i] = L^{l_i} M^{m_i} T^{\tau_i}, \quad i = 0, 1, \dots, n, \quad (24)$$

where the exponents l_i, m_i and τ_i are numbers without dimensions. In general, the number of basic physical quantities in the selected system will be denoted by r (in the case of mechanics $r = 3$). From the complete set of physically independent quantities q_1, q_2, \dots, q_n , it is now convenient to choose a *complete, dimensionally independent* subset of k quantities q_1, q_2, \dots, q_k ($k \leq n$). It is now possible to express each of the remaining $n-k$ physically independent quantities q_{k+1}, \dots, q_n , and the quantity q_0 too, through the dimensions of q_1, q_2, \dots, q_k (which form this dimensionally independent subset) as

$$[q_j] = [q_1^{N_{j1}} q_2^{N_{j2}} \dots q_k^{N_{jk}}], \quad (25)$$

where $j = 0$ and $j = k+1, \dots, n$. (Subset q_1, q_2, \dots, q_k is *complete* if the dimensions of the remaining quantities q_{k+1}, \dots, q_n can be expressed through the dimensions of the quantities q_1, q_2, \dots, q_k and it is *dimensionally independent* if the dimensions of any of its members cannot be expressed through dimensions of other quantities from that subset.) Selecting a subset with these properties is not unique, but the number of dimensionally independent quantities that make it cannot be greater than the number of basic dimensions that appear in the dimensions of the initial set of quantities, i.e. $k \leq r$ must be true. In accordance with such grouping of quantities, relation (23) becomes

$$q_0 = f(q_1, \dots, q_k, q_{k+1}, \dots, q_n). \quad (26)$$

The exponents N_{ji} ($i = 1, \dots, k$) in formula (25) are rational numbers without dimensions that are determined by algebraic methods. The formal procedure will be illustrated with the example of mechanics where the dimensions of all quantities can be shown in formula (24). Let q_1, q_2 and q_3 make a complete dimensionally independent subset. Substituting formula (24) into relation (25), as a result of the request for its dimensional homogeneity, we obtain the following system of algebraic equations:

$$\begin{aligned} l_j &= N_{j1}l_1 + N_{j2}l_2 + N_{j3}l_3 \\ m_j &= N_{j1}m_1 + N_{j2}m_2 + N_{j3}m_3 \\ \tau_j &= N_{j1}\tau_1 + N_{j2}\tau_2 + N_{j3}\tau_3, \end{aligned} \quad (27)$$

¹⁴ So for example, the value of velocity of the object that moves through the fluid will not affect the other relevant quantities: surface of the object, density of the environment and its viscosity.

¹⁵ For example, in the formula $A = Be^{-C-(D_1+D_2)}/E+F$, C must be dimensionless, D_1 and D_2 must have the same dimensions as well as $A, B, D/E$ and F .

where, as already emphasized, $j = 0$ and $j = k + 1, \dots, n$. For each value of the index j or in other words, for each of dimensionally dependent physical quantities q_0, q_{k+1}, \dots, q_n , we get a system of three algebraic equations with three unknowns (N_{j1}, N_{j2}, N_{j3}) that can be easily solved. As both sides of equation (25), for each value of the index j , have the same dimensions, dividing the left-hand side by the right, we get a series of dimensionless monomial:

$$\Pi_0 = \frac{q_0}{q_1^{N_{01}} q_2^{N_{02}} \dots q_k^{N_{0k}}}, \quad (28)$$

$$\Pi_1 = \frac{q_{k+1}}{q_1^{N_{(k+1)1}} q_2^{N_{(k+1)2}} \dots q_k^{N_{(k+1)k}}}, \quad (29)$$

$$\Pi_{n-k} = \frac{q_n}{q_1^{N_{n1}} q_2^{N_{n2}} \dots q_k^{N_{nk}}}. \quad (30)$$

If we still use equation (26), the first of these formulae becomes

$$\Pi_0 = \frac{f(q_1, \dots, q_k, q_{k+1}, \dots, q_n)}{q_1^{N_{01}} q_2^{N_{02}} \dots q_k^{N_{0k}}}. \quad (31)$$

It contains dimensionally dependent quantities (q_{k+1} to q_n) which we can express through the monomials defined by relations (29) and (30) so we get

$$\Pi_0 = \frac{f(q_1, \dots, q_k, \Pi_1 q_1^{N_{(k+1)1}} \dots q_k^{N_{(k+1)k}}, \dots, \Pi_{n-k} q_1^{N_{n1}} \dots q_k^{N_{nk}})}{q_1^{N_{01}} q_2^{N_{02}} \dots q_k^{N_{0k}}}, \quad (32)$$

or

$$\Pi_0 = \phi(q_1, \dots, q_k, \Pi_1, \dots, \Pi_{n-k}), \quad (33)$$

where ϕ denotes the new functional dependence within the set of dimensionally independent quantities k (q_1, q_2, \dots, q_k) which also includes all dimensionless combinations of remaining $n-k+1$ values (q_0, q_{k+1}, \dots, q_n). Since all quantities that appear in the final equation are dimensionless except q_1, \dots, q_k , their values do not depend on the choice of the basic unit system¹⁶. This means that while changing the unit system, the quantity Π_0 does not change, meaning that

$$\frac{\partial \Pi_0}{\partial q_1} = 0, \quad \frac{\partial \Pi_0}{\partial q_2} = 0, \dots, \quad \frac{\partial \Pi_0}{\partial q_k} = 0, \quad (34)$$

whence we conclude that quantity Π_0 in fact does not depend on any of the q_1, \dots, q_k so relation (33) takes the final form

$$\Pi_0 = \Phi(\Pi_1, \dots, \Pi_{n-k}). \quad (35)$$

Since expression (35), within dimensionless monomials Π_0 to Π_{n-k} , contains all the relevant physical quantities (q_1, q_2, \dots, q_n) that describe the observed process, it is actually just a second, more suitable, starting form of equation (23). The formula obtained is the final result of dimensional analysis and represents the so-called Buckingham π theorem¹⁷, which can be formulated like this: let us take q_0 to be a physical quantity the value of which is determined by n independent physical quantities q_1, q_2, \dots, q_n , of which k are mutually dimensionally independent. The remaining $n-k$ quantities can be expressed as dimensionless and independent

¹⁶ Let us recall that these quantities which remained in expression (33) cannot form dimensionless combinations because it was the main criteria for choosing them, i.e. they were chosen to be dimensionally independent.

¹⁷ The theorem was named after the fact that Buckingham has marked dimensionless combinations by symbol Π , in his original paper from 1914.

quantities within relation (35). The number of independent quantities that describe the given physical problem in this case decreases from n to $n-k$ [3, 11]. Dimensionless relations Π_0 to Π_{n-k} are often called Π groups. We note that the Π group denotes grouping of quantities rather than the group in mathematical sense. The main advantage of dimensional analysis is exactly in that. Although while using it, we cannot determine the functional dependence (35), it allows us, by using a relatively simple procedure, to reduce the number of arguments of the starting functional dependence, and thus reduce the problem to further experimental or theoretical investigation of a small number of physically independent dimensionless quantities whose values determine the observed q_0 quantity.

4. The applications of the Buckingham theorem

As we have seen, we get dimensionless relations of physically independent quantities that determine the observed quantity q_0 , by applying the principle of homogeneity, which states, in short, that both sides of the equation that describes a physical phenomenon must have the same dimensions, and by applying the analytical principles, according to which every physical phenomenon is described by an analytical function (23). The procedure of application of the Buckingham theorem can be implemented through the following steps:

- (1) identifying a complete set of physically independent quantities,
- (2) forming of a subset of dimensionally independent quantities,
- (3) constructing the dimensionless-monomial relations (28)–(30),
- (4) representing their final dependence in the form of (35).

4.1. Application of the Buckingham theorem to the analysis of an object's motion through a fluid

The first step stipulated by the application procedure of the Buckingham theorem consists in determining the complete set of physically independent quantities, i.e. in taking into account all quantities which determine the value of the resistance force ($q_0 = F$) with object movement through the fluid. As we have seen, they are object's velocity $q_1 = u$, its cross-sectional area $q_2 = S$, density of the fluid $q_3 = \rho$ and its viscosity determined by the corresponding coefficient $q_4 = \eta$, so that the required set is (u, S, ρ, η) . The number of physically independent quantities n in this case is 4, and the obtained set is complete and physically independent but not unique. Another such set could be (u, L, ρ, η) , where L is a characteristic dimension of the object. In the second step we should make the dimensional analysis of a set of physically independent quantities (u, S, ρ, η) and separate from it the dimensionally independent subset¹⁸. As the dimensions of these quantities are $[u] = LT^{-1}$, $[S] = L^2$, $[\rho] = ML^{-3}$, $[\eta] = ML^{-1}T^{-1}$, one can immediately see that they are not dimensionally independent. However, if we form a subset (u, S, ρ) ,¹⁹ all the quantities will be dimensionally independent inside it. We conclude that, in this case, $k = 3$. It means that we will have only one more monomial apart from Π_0 (since $n-k = 4-3 = 1$), which appears on the right-hand side of formula (35). In the third step it is necessary to determine these monomials. They will be, according to (28) and (29),

$$\Pi_0 = \frac{F}{u^{N_{01}} S^{N_{02}} \rho^{N_{03}}}, \quad (36)$$

¹⁸ The problem is mechanical so the basic physical quantities are length, mass and time, which means that $r = 3$.

¹⁹ Note that a subset of dimensionally independent quantities is not unique. In this case it is possible to select three more subsets: (u, S, η) , (S, ρ, η) and (u, ρ, η) but by a direct check we see that it will not affect the final result.

$$\Pi_1 = \frac{\eta}{u^{N_{41}} S^{N_{42}} \rho^{N_{43}}}, \quad (37)$$

with coefficients which have to be determined from the condition of dimensional homogeneity. The appropriate systems of algebraic equations are obtained based on the relations

$$\begin{aligned} L^0 M^0 T^0 &= L M T^{-2} (L T^{-1})^{-N_{01}} (L^2)^{-N_{02}} (L^{-3} M)^{-N_{03}}, \\ L^0 M^0 T^0 &= L^{-1} M T^{-1} (L T^{-1})^{-N_{41}} (L^2)^{-N_{42}} (L^{-3} M)^{-N_{43}}, \end{aligned}$$

by comparing the left- and right-hand side powers. It is easy to see that for the required coefficients we obtain $N_{01} = 2$, $N_{02} = 1$, $N_{03} = 1$, i.e. $N_{41} = 1$, $N_{42} = 1/2$, $N_{43} = 1$. Based on that expression (36) and (37) become

$$\Pi_0 = \frac{F}{u^2 S \rho}, \quad (38)$$

$$\Pi_1 = \frac{\eta}{u S^{1/2} \rho}, \quad (39)$$

where you can see that the monomial Π_1 equals the inverse value of the Reynolds number (8). Finally, in the fourth step, on the basis of the theorem, i.e. between the two monomials there is a functional relationship in the form of (35), i.e.

$$\frac{F}{u^2 S \rho} = \Phi \left(\frac{\eta}{u \sqrt{S} \rho} \right). \quad (40)$$

Given the complete arbitrariness of this functional dependence, it can be written in the form of (11).

4.2. Application of the Buckingham theorem on propagation of waves on water

Let us now apply the Buckingham theorem on propagation of waves on water. The complete set of physically independent quantities that determine the value of speed of propagation of waves ($q_0 = u$) are the density of water $q_1 = \rho$, wavelength $q_2 = \lambda$, the strength of the gravity field $q_3 = g$ and surface tension $q_4 = \sigma$, which means that the number of physically independent quantities is again $n = 4$. Dimensions of the abovementioned, physically independent quantities are $[\rho] = M L^{-3}$, $[\lambda] = L$, $[g] = L T^{-2}$ and $[\sigma] = M T^{-2}$. From the set $(\rho, \lambda, g, \sigma)$, we extract one subset of dimensionally independent quantities. Let this be the subset (ρ, λ, g) . Since $k = 3$ the number of monomial on the right-hand side of expression (35) is $n - k = 4 - 3 = 1$. This means that it is necessary to form and determine a total of two dimensionless forms Π_0 and Π_1 . Forms of these monomials, in accordance with (28) and (29), will be

$$\Pi_0 = \frac{u}{\rho^{N_{01}} \lambda^{N_{02}} g^{N_{03}}}, \quad (41)$$

$$\Pi_1 = \frac{\sigma}{\rho^{N_{41}} \lambda^{N_{42}} g^{N_{43}}}, \quad (42)$$

Based on the principle of dimensional homogeneity, we obtain

$$\begin{aligned} L^0 M^0 T^0 &= L T^{-1} (L^{-3} M)^{-N_{01}} (L)^{-N_{02}} (L T^{-2})^{-N_{03}}, \\ L^0 M^0 T^0 &= M T^{-2} (L^{-3} M)^{-N_{41}} (L)^{-N_{42}} (L T^{-2})^{-N_{43}}. \end{aligned}$$

Solving the corresponding system of equations for the unknown coefficients, we obtain $N_{01} = 0$, $N_{02} = 1/2$, $N_{03} = 1/2$ and $N_{41} = 1$, $N_{42} = 2$, $N_{43} = 1$ so the final form of expression (41) and (42) will be

$$\Pi_0 = \frac{u}{\sqrt{\lambda g}}, \quad (43)$$

$$\Pi_1 = \frac{\sigma}{\lambda^2 \rho g}. \quad (44)$$

Π_1 monomial is therefore equal to the reciprocal value of the Bond number (19). According to the Buckingham theorem, the dependence $\Pi_0 = \Phi(\Pi_1)$ now has the final form

$$\frac{u}{\sqrt{\lambda g}} = \Phi\left(\frac{\sigma}{\lambda^2 \rho g}\right), \quad (45)$$

which can be reduced to the form (22) with minimal transformation.

5. Dependence on dimensionless Π monomials

As already noted, the functional dependence in expression (35) is, for a specific case, usually determined experimentally. Thus, in the case of an object's motion through the fluid, we need to, in some way, determine the value of the δ degree in expression (10), i.e. functional dependence $\Pi_0 = \Phi(\Pi_1)$ presented by the general relation (35) or in this particular case (40). We can immediately note that for $\delta = 1$, relation (10) gives formula (5) for force, while in the case when $\delta = 0$, the force has the form of (2). It is actually about the limiting cases, i.e. the corresponding asymptotic behaviour of the function Φ in the expression $\Pi_0 = \Phi(\Pi_1)$. According to [12], if we assume that $\Pi_1 < 0$, then the quantity Π_0 *essentially depends on* Π_1 if this is true

$$\lim_{\Pi_1 \rightarrow 0} \Phi(\Pi_1) = \begin{cases} 0 \\ \infty \end{cases}. \quad (46)$$

We note that this assumption will not diminish the generality of the conclusion because the shape of the function in relation (35) is completely arbitrary, which means, among other things, that arbitrary algebraic transformations are allowed, including taking the reciprocal value. The words *depends essentially* mean that the final form of expression $\Pi_0 = \Phi(\Pi_1)$ a dimensionless monomial will need to appear in such a way that will affect its ultimate functional dependence. In this case it is necessary to observe relation (40) which we will write in the form

$$F = u^2 S \rho \Phi(Re). \quad (47)$$

The experiments show that in two borderline cases the function $\Phi(Re)$ has the values [13]

$$\Phi(Re) \propto \begin{cases} 1/Re, & Re < 10 \\ 1, & Re > 100 \end{cases}. \quad (48)$$

The range $10 < Re < 100$ is a transition region.

In other words, if the flow is such that the Reynolds number is less than 10, relation (47) becomes

$$F \propto u^2 S \rho \frac{1}{Re} = \eta u L. \quad (49)$$

We obtained this formula as a result of the fact that during such a regime of motion of the object, we have essential dependence on the shape of monomial which represents the Reynolds number. In another limiting case, we get the following expression for force:

$$F \propto u^2 S \rho. \quad (50)$$

Thus, the Reynolds number, during this regime of movement, does not essentially change the dependence of force from the one we got by dimensional analysis. It is easy to show that the

situation is very similar to the formulae of the speed of waves on water. Namely, expression (21) for $\delta = 1/2$ comes down to relation (16), while for $\delta = 0$ it becomes relation (13). If we look through the Buckingham theorem again, we again deal with the corresponding behaviour of functional dependence $\Pi_0 = \Phi(\Pi_1)$ in areas where it has either the value of $1/\sqrt{Bo}$ or value 1.

6. Conclusion

As we have seen, there are two approaches to solving the problem using dimensional analysis, the first called *intuitive* and the other based on the *Buckingham theorem*. The first approach is usually applied when the number of physical quantities considered is relatively small (3–4), and if there are more, it is more rational to apply the Buckingham theorem.

Dimensional analysis is a powerful and universal tool to understand the properties of physical quantities independent of the units used to measure them. Therefore, it is not surprising that the first ideas about its use can be found in Fourier [4], and that it was applied in works of Maxwell, Rayleigh, Einstein, Planck, etc. The complete theoretical framework of dimensional analysis is given by the Buckingham theorem, the help of which the dependence of the observed physical quantity q_0 on the often large number of other quantities (q_1, q_2, \dots, q_n) can be reduced to the minimal, or invariant, dependence between the so-called Π groups. These groups are dimensionless numbers which are often called *similarity numbers* or *scaling invariants* of the given physical problem (Reynolds, Bond, Mach, and other numbers). This is easily understood in the case of the Reynolds number, which is very commonly found as an example of the Π group. Suppose we want to determine how a type of aircraft will behave when it moves through air flowing around it in a certain way. For practical reasons it is better to perform the corresponding experiments on reduced models of the aircraft for which the Rayleigh number will have the same value as in the case of movement of the real aircraft. Only in this case will the conclusions that follow from the experiment with the model aircraft be valid for the original system also, as both are described by the same dimensionless model. In other words it means that we can be sure that the effects that occur will be equal to those that will occur during the actual aircraft flight. Let us point out a few more important facts which we encounter in dimensional analysis. The most important thing is proper selection of the complete set of physically independent quantities. Their choice is not trivial and it must be guided by certain principles. Among them the so-called principle of continuity is important, which actually represents a fact that small quantities usually have little impact on the monitored system. Although we know that in complex systems there are deviations from this rule (the butterfly effect of chaos theory), it is still a rule that applies in most cases. If it were not so, it would be impossible to predict the results of experiments because they would be affected by a large number of factors (physical quantities) present in nature, whose values we could not determine in any way. Common idealizations we commonly use in physics, such as object movement without friction, inextensible string, material points ideal resistors and capacitors, precisely find the justification in the principle of continuity. In an attempt to construct an idealized physical model of the real physical situation, we are always forced to make certain approximations. Good knowledge of physical processes in a given system gives us the right not to take into consideration some of the quantities, considering that under given conditions they have negligible effects in the formation of a complete set of physically independent quantities. Finally, let us note that within this approach, we cannot obtain the values of dimensionless constants of proportionality that appear in formulae. As a rule, however, these constants have values $1/2, \sqrt{2\pi}, 1/\sqrt{2\pi} \dots$, meaning that they are neither too big nor too small, so that dimensional analysis can also serve to estimate the order of magnitude of physical quantities.

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